How science comprehends chaos

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Behaviors of chaotic systems are unpredictable. Chaotic systems are deterministic, their evolutions being governed by dynamical equations. Are the two statements contradictory? They are not, because the theory of chaos encompasses two levels of description. On a higher level, unpredictability appears as an emergent property of systems that are predictable on a lower level. In this talk, we examine the structure of dynamical theories to see how they employ multiple descriptive levels to explain chaos, bifurcation, and other complexities of nonlinear systems.

Levels of description

Simplicity may have a unified form, but complexity has many varieties. The ability to adopt various intellectual focuses and perspectives suitable for various topics is essential to the study of complexity. Hence, a striking feature of the sciences that wrestle with complex phenomena is the diversity of theoretical perspectives, models, and levels of description. Two kinds of levels are prominent:

- Scope or generality: levels of generality are related by class inclusion: membership of a more general level includes membership of a less general level, as the class of animals includes the class of mammals, which includes the class of humans. Levels of higher generality have larger scopes but less details; general descriptions cover more members but leave out some peculiarities of individual members.

- Organization: levels of organization are related by composition: entities of a higher organizational level is composed of entities on a lower organizational level, as an organism is composed of organs, and an organ composed of cells.

Different levels exhibit different phenomena, and sometimes the differences are huge. For instance, iron rods are rigid; the rigidity and flexibility of various materials have been scientifically measured and studied. However, the iron atoms that make up the rod are anything but rigid; atoms are ghostly quantum mechanical entities unlike anything familiar in our world of medium-sized things.

Faced with phenomena so different, theories and descriptions on different levels employ different concepts, which are sometimes incompatible or contradictory. How do concepts and theories on various levels relate to each other? At least three philosophical attitudes exist.

Reductionism insists on the dictatorship of a single level, so that all descriptions of the world should be the logical consequences of theories on the chosen level. Other levels are in principle superfluous.

Parochialism acknowledges different descriptive levels but denies any rational connection between them. Without relation, the notion of levels loses significance, leaving behind various incommensurate paradigms.

Synthetic analysis accepts both the multiplicity of descriptive levels and the possibility of connecting them, admitting that the connections would be neither easy nor as perfect as those demanded by reductionism would. Inexact connections between different levels engender the notion of *emergent property*, which is excluded from reductionism and parochialism.

Synthetic analysis is more difficult to argue for than the black-or-white doctrines of reductionism and parochialism. It occupies grey areas and appeals to pragmatic concerns. Fortunately, nonlinear dynamics offers a good case to study how it works. Nonlinear dynamics is a major tool in the research on complex systems. Its concepts of chaos, bifurcation, instability, and strange attractor are popularized and sometime hyped. Nevertheless, being a mathematical theory, its rigorous conceptual structures provide a clear picture of how scientists represent and explain complex phenomena.

Deterministic chaos

Let us start with chaos. The butterfly effect, by which the flutter of wings of a butterfly in Nicaragua leads to a tornado in Texas, is a case of chaos. Yet this exotic effect, widely popularized in *Jurassic Park* and other media, belongs to the same general class as prosaic planetary motions. Both are dynamic systems. (By the way, recent research shows that the Earth's trajectory is also chaotic, although is a way less dramatic to excite movie directors).

Dynamics is a staple of physics. Dynamical processes or the evolutions of dynamical systems are governed by dynamical equations. They are deterministic. Being *deterministic* means that for every stage in a dynamical process, the dynamical equation determines a unique successor stage; equivalently, for a dynamical system's state at every time, the dynamical equation determines a unique state for the next time. Thus given an initial condition, i.e., the system's state at an initial time, the dynamical equation predicts the system's subsequent behaviors. (A system's state is the summary of its characteristics).

As an example, consider l*ogistic systems* governed by the dynamical equation:

$$x_{n+1} = ax_n - ax_n^2 \ .$$

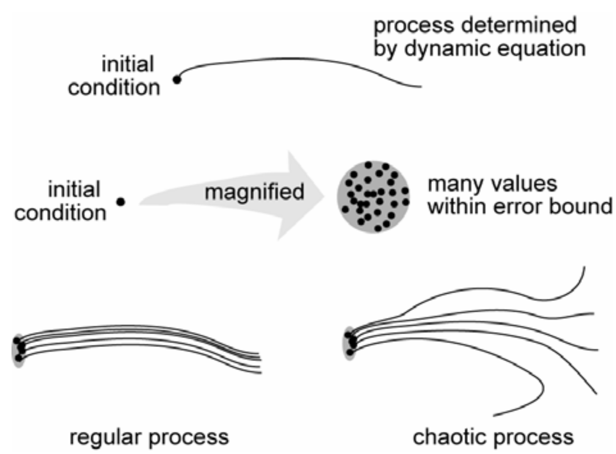
This equation is *nonlinear* because of the x_n^2 term. And you have guessed it, it can engender chaotic behaviors. This simple equation has many applications, e.g., to represent population growth in ecology. Here the independent variable *n* is usually interpreted as time, "discrete time" in which the system is measured at regular intervals, such as every second or every day. The state variable x_n represents how the system's state *x* changes with respect to *n*; e.g., x_0, x_1, x_2

respectively describes the system at midnight, 1 am, 2 am. The third element in the equation, a, describes certain "fixed" characteristics of the system. We will return to it later, now it suffices to note that a can be assigned any value, and once assigned, the value remains constant during the dynamical process.

The logistic process is obviously deterministic. Given a value for a and an initial condition x_0, the logistic equation determines the values of all sequent x_n. You pluck the value of x_0 it in the right-hand side of the equation and calculate the value of x_1. Then you put x_1 in the right-hand side, repeat, to generate the dynamic process. The iterative calculation is trivial. It makes a good arithmetic exercise for kids. One expects a class of kids to return the same answers for the same values of a and x_0. They usually do, but not always.

If a teacher happens to pick the values $a = 4$ and $x_0 = 0.3$, consensus would not last long. The class would quickly plunge into chaos as kids come up with widely different answers for large n, not because their arithmetic is wrong or because their hand calculators malfunction, but because for $a = 4$ and most initial conditions $0 < x_0 < 1$, the logistic system is chaotic.

Chaos in mathematics means extreme sensitivity to initial conditions, i.e., minute differences in the initial conditions are amplified exponentially. "Initial" is relative term; we can regard any x_n as "initial" and consider the process from then on. The kids all start with the same x_0, but if some kids round off to 2 decimal places in their calculation and others to 3, they would get slightly different values for x_3. This difference is amplified from then on to yield grossly disparate answers. Of course, the kids can calculate more accurately, say to six or sever decimal places, but as long as the calculation is not *absolutely* accurately, eventually divergence occurs.



Given an initial condition, the dynamic equation determines the dynamic process, i.e., every step in the evolution. However, the initial condition, when magnified, reveals a cluster of values within a certain error bound. For a *regular* dynamic system, processes issuing from the cluster are bundled together, and the bundle constitutes a predictable process with an error bound similar to that of the initial condition. In a *chaotic* dynamic system, processes issuing from the cluster diverge from each other exponentially, and after a while the error becomes so large that the dynamic equation losses its predictive power.

The demand for absolute accuracy, in initial condition specification or in calculation, has great consequence for natural science, because it is impossible to meet. Most important physical quantities are continuous and represented by *real numbers* with uncountably infinitely many values. To specify a quantity with absolute accuracy requires a number with infinitely many decimal places, this *infinite* amount of information is beyond human capability. Computers, no matter how powerful, are necessarily *finite* state machines. Inconsistency in repeated computer runs of weather models first alerted meteorologists to chaos.

Real number representation and the inaccuracy it implies are ubiquitous in natural science, not the least physics. Why is science so successful in predicting physical phenomena? Scientists are lucky because many natural processes are not *chaotic* but regular.

Because you are unable to muster infinite amount of information, any initial condition you give is actually an infinite number of conditions clustered within a certain margin of error. For regular systems, the processes ensuing from the conditions tend to bundle together, so that the dynamic equation can predict their evolutionary courses to within a similar margin of error. For chaotic systems, the processes may bundle together for a short while, but will eventually diverge and diverge big time. Although the dynamic equation determines each step uniquely, it loses its predictive power over the long run, because the answers spread all over the place. Chaos and long-term unpredictability are *emergent* properties of dynamical processes.

A dynamic process is made up of successive stages. As a characteristic of dynamic processes, chaos exemplifies several features of emergent properties:

1. It is absent in the process's constituent stages or the mere sum of the stages. Worse, it is not only absent there, it cannot possibly be there because it runs against their definitions.

2. It is the property of dynamical processes as wholes, defined on the process level by comparing the behaviors of various processes. Compared to the characteristics of the constituent stages, it is a very different type of property.

3. It becomes important in the long run, when a process has accumulated many stages. A long process is a large temporally extended system, and emergent chaos is important for large systems.

4. Chaos is explicable; there is a rigorous mathematical theory for it. The explanation is not reductive. To define chaos, we need to expand the scope of generalization to include various initial conditions and the divergence between various processes. This is possible only from a high-level perspective where we can grasp and compare different processes as wholes. This is the perspective of synthetic analysis.

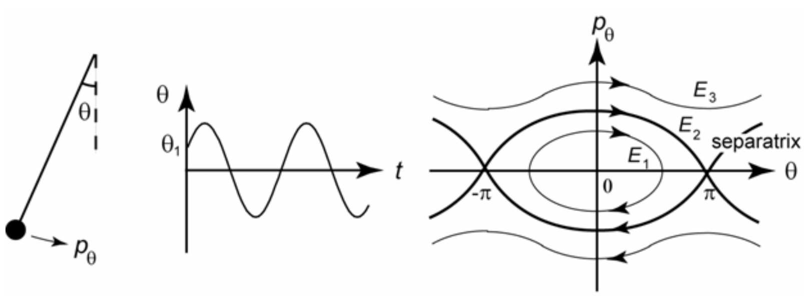
A synthetic framework for analysis

Chaos and unpredictability are often used by postmodernists in attacking science. Chaos does reveal a basic limitation to what science can achieve. However, contrary to postmodernism, to know one's own limitation is not unscientific but essential to science. It is also important to see how the self-knowledge is attained, and what capacities are made possible in the process.

Dynamics is as old as Newton. However, classical dynamics mainly focuses on individual processes obtained by solving dynamic equations with specific initial conditions. A process is nothing but the temporal succession of consecutive steps, each determined by its predecessor. Classical dynamics lacks the conceptual means to represent phenomena such as chaos or bifurcation. What makes modern dynamics more powerful is the global geometric view introduced by Henri Poincare at the end of the last century.

Let us compare how classical and modern dynamics treat an undamped and undriven pendulum. The pendulum's state at any time is specified by two quantities, its angular displacement θ and its angular momentum p_θ. Given an initial condition specified by a displacement θ_1 and an energy E_1, the dynamic equation predicts how the pendulum's displacement θ undulates as it swings back and forth, stopping and turning at a maximum displacement determined by E_1. The temporal evolution, which constitutes a dynamic process with the initial condition, is the whole point of classical dynamics.

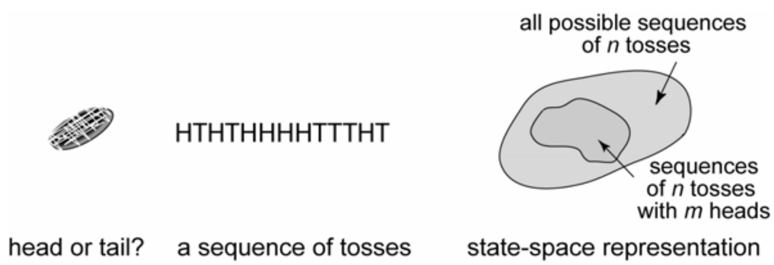
Modern dynamics goes much further. It introduces an expansive conceptual framework that includes processes for *all* possible initial conditions, and if a system depends on a set of parameters, its behaviors for all values of the parameters. The framework summarizes all these possible behaviors by a portrait in the system's *state-space* or *phase space*. The *state* of a dynamic system is a summary of all its properties at one moment of time, and a process is a sequence of states whose succession is governed by the dynamic equation. The state space is the collection of all the states that a system can possibly achieve. It has become one of the most important concepts in all mathematical sciences.



The pendulum's two variables, θ and p_θ, span its state space. As the pendulum swings, its state traces out an ellipse in the state space; its displacement reaches a certain maximum when is momentum vanishes, and its momentum is maximum when it is not displaced from the vertical. The maximum is determined by the pendulum's energy E. Another process starting from an initial condition with higher energy traces a larger ellipse. When the pendulum's energy E_2 is great enough for it to go over the top, its behavior changes radically. At energy above E_2 it no longer swings but rotates about its pivot, and its momentum never vanishes. The state space portrait of the pendulum reveals two distinct *types* of motion, swings and rotations, separated by the singular case where the pendulum precariously stops at the top.

The singular case, which separates the two general types, is called a *separatrix* in the state space. For dissipative systems, the separatrix would separate two basins of attraction, and the ellipse would be an *attractor*.

In sum, modern dynamics does not lose sight of individual states or stages in a process, so that we can analyze the processes if we please. However, it also introduces an expanded conceptual framework that encompasses *all possible processes* for a type of system, defines properties of the processes as wholes, and systematically classifies the processes according to their properties. If solving for individual dynamical processes is like studying individual trees, the state-space representation is like an aerial view of the forest. The aerial view reveals patterns not visible on ground. Discerning patterns – *types* of properties – and introducing concepts to represent them systematically are central to scientific research. Such synthetic analytic conceptual frameworks are not limited to modern dynamics. They are also the secrets to the probability calculus.



The probability calculus has found application in so many areas historians talk about a "probability revolution." What in the probability calculus makes it so powerful? No, not chance; the notion of chance is not even defined in the calculus. What makes the probability calculus powerful is the synthetic framework for representing a large composite system as a whole that is susceptible to analysis, for example, to treat a sequence of coin tosses as a unit, and represent all possible configurations of the sequence in a state space, often called a probability space. Instead of considering the time variation as in dynamics, the probability calculus introduces a systematic way to partition the state space into chunks and calculate the relative magnitudes of the chunks. The relative magnitudes are defined as probabilities. For example, the ratio of the two shaded areas in the diagram gives the *probability* of getting m heads in n tosses of a coin ($m \leq n$).

Further generalization: attractors and bifurcation

Dynamical equations are already generalizations; each equation governs a class of dynamical systems. To solve the equation requires an initial condition. Classical dynamics dose not generalize over initial condition, whose value for each case is assigned "from outside." Modern dynamics generalizes over initial conditions by making it a *theoretical variable* internal to the state-space representation. This broadened theoretical framework enables scientists to introduce new concepts for dynamical processes with various initial conditions. Chaos and attractor are such novel concepts.

By internalizing initial conditions instead of receiving them "from outside," dynamical theory attains a higher level of generality. Its scope expands from individual processes to processes with all possible initial conditions. Further generalization and scope expansion are on the way. The state-space depicted above encompasses trajectories with all possible energies for a pendulum with a certain physical configuration. What happens if we change the pendulum's length? So far it must be changed "by hand" in the dynamic equation, but can we gain more insight by making length into a variable?

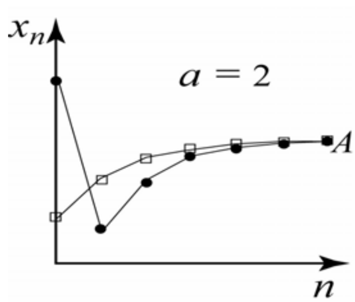
Let us return to the logistic equation $x_{n+1} = ax_n - ax_n^2$ and look at the parameter a. It is called a "control parameter," the value of which is "dialed in by hand." Because control parameters are totally enmeshed in the structure of the equation, to vary them is trickier and not always fruitful. We cannot simply let a vary with n, for that would result in a totally different dynamical equation. Luckily, logistical systems have certain peculiarities that make the variation of a meaningful, provided we accept certain approximations.

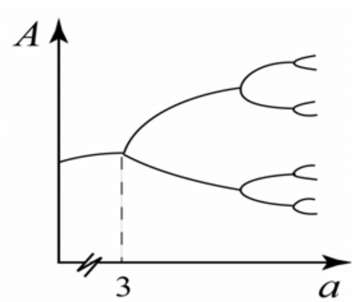
Suppose we pick a value for a and calculate processes for various initial conditions. For, say, $a = 2$, we find that no matter what value we give for x_0, x_n always ends up at 0.5 when n becomes large; for $a = 1.5$, x_n always end up at 0.33, and so on. This value, (or group of values,) to which processes initiating from different conditions converge, is called an *attractor*. It is denoted by A in the diagram.

Dynamical system $f(x_n, a)$

	State variable x_n	Control parameter a
Variation	governed by f	independent of f
Classification	basins of attraction	systems of basins
Instability	chaos	bifurcation
Characteristic time	*T1*	*T2 >> T1*
Examples:		
ecology	population density	environmental change
olfaction	separation of odors	development

logistic systems
$x_{n+1} = ax_n(1 - x_n)$





Logistical processes have an attractor A, whose behaviors changes with a, not only quantitatively but qualitatively. For $a < 3$, A has a single value, which increases gradually as a increases.

Then, at $a = 3$, the attractor becomes unstable, beyond 3 it changes into a cycle of two values. For $a = 3.1$, for instance, $x_n = 0.56$, $x_{n+1} = 0.76$, $x_{n+2} = 0.56$, $x_{n+3} = 0.76$, and so on, jumping between two steady values. As a increasing further, the attractor changes into a cycle of four values, then cycle of eight values, until at $a = 4$ the attractor has infinitely many values and beyond my ability of drawing.

The change in the qualitative pattern of attractors is called *bifurcation*, a novel concept made possible by generalization over control parameters.

Unlike the logistical system, which has only one attractor, many systems have several attractors. The state space of such a dynamical system divides into many *basins of attraction* separated from each other by separatrices that, like continental divides, separate a continent into several drainage basins. Processes initiating from conditions that fall within an attractor basin all converge on the same attractor, just as rains dropping in a drainage basin all converge on the same river.

Attractor and bifurcation are not merely mathematical curiosities; they play significant roles in the sciences. An attractor represents the long term and steady behavior of a dynamical system, when perturbations represented by various initial conditions die down. Such steady behaviors are interesting to scientists.

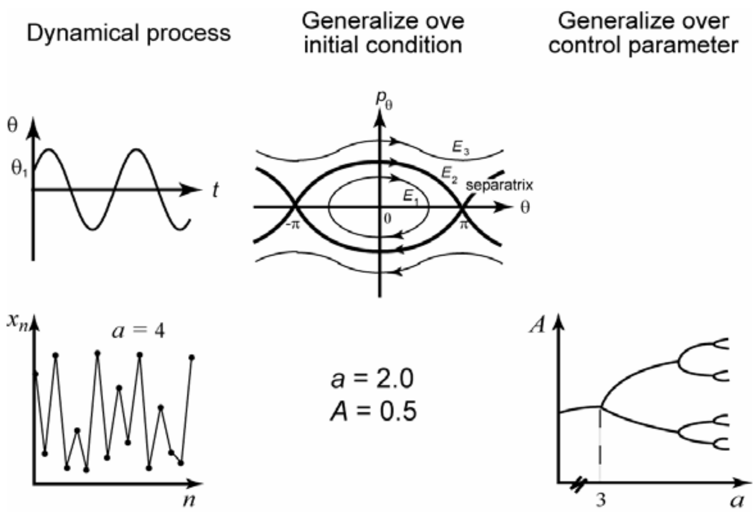
For instance, a dynamical system with many attractors can represent the cognitive process by which an animal distinguish odors. Initial conditions to the system represent stimuli to the nose. Stimuli come in infinite varieties, depending on complicated source and environmental conditions. Because of cognitive dynamics, they separate and settle into different attractors, each representing a different odor. The animal is aware not of the dynamics but only of the final attractor, or the odor that it recognizes. However, cognitive scientists are most interested its unconscious cognitive processes.

As an animal grows up, it may develop keener smell and the ability to distinguish more odors. Such development can be represented by bifurcations, or changes in the pattern of attractor basins. Like geological upheavals changing the landscape and creating more drainage basins, new attractor basin may appear in the animal's cognitive dynamics. In such models, the control parameters whose changes engender bifurcation can represent the animal's growing physiological conditions.

Of course, one must be careful to talk about changes in control parameters. Mathematically, the control parameters must be constants in the solution of a dynamical equation; otherwise, we get a different equation. However, we can approximately consider two time scales, one much longer than the other. Smelling an odor takes a few seconds, developing a better sense of smell takes months if not years. Thus even when an animal is growing, the growth rate is slow we can regard the control parameters of its cognitive dynamics to be constants for the duration of the smelling process. It is an approximation, but approximations based on reasonable scales of magnitude prove to be fruitful in the sciences.

Generalization, scope, organization

To summarize, we have looked at three levels of generality in dynamics: the level of individual dynamical process, the generalization over initial condition, and the generalization over control parameter. Each step of generalization internalizes a quantity previously fixed externally by turning it into a theoretical variable, thus expanding the scope of dynamical theory and enabling the discovery of higher-order patterns and regularities.



By making initial condition a theoretical variable, we proceed from individual dynamical processes to state-space representations of all processes and discover chaos and attractors. By making control parameter a theoretical variable, we proceed from attractors to systems of basins of attraction and discover bifurcation.

For broader scope, generalization pays the price of specific details. The state-space representation encompasses all possible dynamical processes at the price of leaving out information about *when* a dynamic process is at a specific stage. In talking about attractors we leave out how individual process proceeds to the attractor. To investigate bifurcation of attractors, we must adopt a long time scale in which short-term variations are invisible. We can recover the neglected information, but then we have to switch back to the level of individual processes, where high-level phenomena such as bifurcation are absent. Each level is superior for explaining certain phenomena. None is able to the cover everything.

Earlier I mentioned two hierarchies of levels: generality and organization. So far, I have focused on levels of generality. However, we can also regard modern dynamics as providing concepts for various levels of *organization*. To see this, let us adopt a four dimensional view of the world by incorporating temporal "extension." Then a dynamical process appears as a composite whole made up of temporal parts or stages, the relation between two successive stages being determined by the dynamical equation. For example, a logistic process is an entity made up of successive stages, (x_0, x_1, x_2, . . , x_{n-1}, x_n, x_{n+1} . . .), where $x_{n+1} = ax_n (1 - x_n)$.

In this view, each stage becomes an entity, a constituent of a larger entity, the dynamical process. Classical dynamics operates on the level of individual stages and their succession. The state-space representation of modern dynamics, which grasps processes as wholes, theorizes on a higher organization level. In this interpretation, modern dynamics becomes a case for the part-whole relationship and the tradeoff in descriptions of wholes and parts.

Far from being incommensurate, various levels of generalization and organization are connected rationally. Some connections are mathematically rigorous, but some fall short of demands of reductionism. The necessity of adopting two disparate time scales for studying dynamics and bifurcation is a case in point. Yet even this is supported by reasonable physical arguments. The existence of multiple levels of generality and multiple intellectual perspectives is not a weakness but strength of science. It enables scientists to attack complex phenomena from many approaches. Dynamical theory shows how different perspectives can be encompassed, not absolutely but rationally, in a federal unity of science.

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http://www.creatingtechnology.org/essays/chaos.htm